Chapter 1

Frequency response analysis of feedback control systems

1.1 Introduction

With frequency response – using Bode plots – we can analyze dynamic properties of feedback control systems. These properties refers to

- dynamic setpoint tracking, and
- dynamic disturbance compensation.

By definition, in frequency response analysis all signals in the system are assumed to be sinusoids. This seems to limit the usefulness of such analysis because in real systems signals are rarely sinusoids. Still, the frequency response analysis provides useful insight about the dynamic properties of a control system because signals are varying more or less, i.e. the signals can be said to have certain frequency components.

The frequency response analysis is based on certain transfer functions of the control system, namely

- the tracking transfer function, \( T(s) \), and
- the sensitivity transfer function, \( S(s) \).

This analysis assumes a linear model of the control system. However, practical control systems are nonlinear due to phenomena as saturation,
hysteresis, stiction, nonlinear signal scaling etc. Such nonlinearities can influence largely the dynamic behaviour of the control system. To perform “linear” analysis of a non-linear model, this model must be linearized about some operating point. Thus, the results of the analysis will be valid at or close to the operation point where the linearization was made. This fact limits the usefulness of a theoretical analysis of a given nonlinear control system using linear systems methods, but the results may still be useful, particularly if the system most of the time operates close to the chosen or specified operating point.

Although a “linear” analysis of a given nonlinear control system may have limited value, you will get much general understanding about the behaviour of control systems through analysis of examples of linear control systems.

Note: Once you have a mathematical model of a given control system, you should definitely run simulations as a part of the analysis. This applies for both linear and nonlinear control systems. Actually, you may get all the answers you need by just running simulations. The types of answers may concern response-time, static control error, responses due to process disturbances and measurement noise, and effects of parameter variations.

1.2 Definition of setpoint tracking and disturbance compensation

Figure 1.1 shows a principal block diagram of a control system. There are
two input signals to the control system, namely the setpoint $y_{SP}$ and the disturbance $v$. The value of the control error $e$ is our primary concern (it should be small, preferably zero). Therefore we can say that $e$ is the (main) output variable of the control system. The value of $e$ expresses the performance of the control system: The less $e$, the higher performance. $e$ is influenced by $y_{SP}$ and $v$. Let us therefore define the following two concepts:

- The **setpoint tracking property** of the control system concerns the relation between $y_{SP}$ and $e$.
- The **disturbance compensation property** of the control system concerns the relation between $v$ and $e$.

Totally, the setpoint tracking and disturbance compensation properties determine the performance of the control system.

### 1.3 Definition of characteristic transfer functions

#### 1.3.1 The Sensitivity transfer function

We assume that the control system has a transfer function-based block diagram as shown in Figure 1.2. In the block diagram $U_0(s)$ represents the Laplace transform of the nominal control variable $u_0$. In the analysis we will set $u_0$ to zero, because we will focus on the responses of the control system due to the setpoint and disturbance inputs.

We regard the setpoint $y_{SP}$ and the disturbance $v$ as input variables and the control error $e$ as the output variable of the system. Thus, we will derive the transfer function from $y_{SP}$ to $e$ and the transfer function from $v$ to $e$. From the block diagram we the can write the following expressions for $e(s)$:

$$
e(s) = \frac{1}{H_m(s)} e_m(s) \quad (1.1)$$

$$= \frac{1}{H_m(s)} [y_{m_{SP}}(s) - y_m(s)] \quad (1.2)$$

$$= \frac{1}{H_m(s)} [y_m(s)y_{SP}(s) - H_m(s)y(s)] \quad (1.3)$$

$$= y_{SP}(s) - y(s) \quad (1.4)$$

$$= y_{SP}(s) - [H_v(s)v(s) + H_u(s)H_c(s)e_m(s)] \quad (1.5)$$

$$= y_{SP}(s) - [H_v(s)v(s) + H_u(s)H_c(s)H_m(s)e(s)] \quad (1.6)$$
In (1.6), $e(s)$ appears at both the left and the right side. Solving for $e(s)$ gives

$$e(s) = \frac{1}{1 + H_c(s)H_u(s)H_m(s)} [y_{SP}(s) - H_v(s)v(s)]$$

$$= \frac{1}{1 + L(s)} [y_{SP}(s) - H_v(s)v(s)]$$

$$= S(s) [y_{SP}(s) - H_v(s)v(s)]$$

$$= S(s)y_{SP}(s) - S(s)H_v(s)v(s)$$

$$= e_{SP}(s) + e_v(s)$$

which is a transfer functions based model of the control system. $S(s)$ is the sensitivity transfer function:

$$S(s) = \frac{1}{1 + L(s)}$$

where

$$L(s) \equiv H_c(s)H_u(s)H_m(s)$$

is the loop transfer function which is the product of the transfer functions in the loop. From (1.9) we can calculate the control error for any setpoint.
signal, any disturbance signal and any nominal control signal (assuming we know their Laplace transform).

In the following we discuss the various terms in (1.10).

- **The response in the error due to the setpoint**: The response in the control error due to the setpoint is

\[
e_{SP}(s) = S(s)y_{SP}(s) = \frac{1}{1 + L(s)}y_{SP}(s)
\]

(1.13)

which gives a quantitative expression of the *tracking property* of the control system. The static tracking is given by static error when \(y_{SP}\) is constant. This error can be calculated as follows:

\[
e_{SP} = \lim_{t \to \infty} e_{SP}(t) = \lim_{s \to 0} s \cdot e_{SP}(s)
\]

(1.14)

\[
= \lim_{s \to 0} s \cdot S(s)y_{SP}(s) = \lim_{s \to 0} s \cdot \frac{S(s)y_{SP}(s)}{s} = S(0)y_{SP}(0)
\]

(1.15)

Roughly speaking that the tracking property of the control system are good if the sensitivity function \(N\) has small (absolute) value – ideally zero.

- **The response in the error due to the disturbance**: The response in the control error due to the disturbance is

\[
e_v(s) = -S(s)H_v(s)v(s) = \frac{-H_v(s)}{1 + L(s)}v(s)
\]

(1.16)

which expresses the *compensation property* of the control system. The static compensation property is given by

\[
e_{v_s} = \lim_{t \to \infty} e_v(t) = \lim_{s \to 0} s \cdot e_v(s)
\]

(1.17)

\[
= \lim_{s \to 0} s \cdot [-S(s)H_v(s)v(s)]
\]

(1.18)

\[
= \lim_{s \to 0} s \cdot \left[ -S(s)H_v(s)\frac{v_s}{s} \right]
\]

(1.19)

\[
= -S(0)H_v(0)v_s
\]

(1.20)

From (1.20) we see that the compensation property is good if the sensitivity function \(S\) has a small (absolute) value (close to zero).

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\(^1\)Here the Final Value Theorem of the Laplace transform is used.
1.3.2 The Tracking transfer function

The tracking transfer function \( T(s) \) – or simply the tracking function – is the transfer function from the setpoint \( y_{SP} \) to the process output variable \( y \):

\[
y(s) = T(s)y_{SP}(s)
\]  \hspace{1cm} (1.21)

From the block diagram in Figure 1.2, or by setting \( e_{y_{SP}}(s) \equiv y_{SP}(s) - y(s) \) for \( e_{y_{SP}}(s) \) in (1.13), we can find the tracking function \( T(s) \) as the transfer function from \( y_{SP} \) to \( y \):

\[
\frac{y(s)}{y_{SP}(s)} = T(s) = \frac{H_c(s)H_u(s)H_m(s)}{1 + H_c(s)H_u(s)H_m(s)} = \frac{L(s)}{1 + L(s)} = 1 - S(s)
\]  \hspace{1cm} (1.22)

The static tracking property is given by the static tracking ratio \( T(0) \):

\[
y_s = \lim_{t \to \infty} y(t) = \lim_{s \to 0} s \cdot y(s)
\]  \hspace{1cm} (1.23)

\[
y_s = \lim_{s \to 0} s \cdot T(s)y_{SP}(s) = \lim_{s \to 0} s \cdot T(s)\frac{y_{SP}(s)}{s}
\]  \hspace{1cm} (1.24)

\[
y_s = T(0)y_{SP}
\]  \hspace{1cm} (1.25)

The tracking property is good if the tracking function \( T \) has (absolute) value equal to or close to 1 (since then \( y \) will be equal to or close to \( y_{SP} \)).

In some contexts it is useful to be aware that the sum of the tracking function and the sensitivity function is always 1:

\[
T(s) + S(s) = \frac{L(s)}{1 + L(s)} + \frac{1}{1 + L(s)} \equiv 1
\]  \hspace{1cm} (1.26)

1.4 Frequency response analysis of setpoint tracking and disturbance compensation

1.4.1 Introduction

Frequency response analysis of control systems expresses the tracking and compensation property under the assumption that the setpoint and the disturbance are sinusoidal signals or frequency components of a compound signal. The structure of the control system is assumed to be as shown in Figure 1.2. The Laplace transformed control error is given by (1.9), which is repeated here:

\[
e(s) = S(s)y_{SP}(s) - S(s)H_c(s)v(s)
\]  \hspace{1cm} (1.27)
where $S(s)$ is the sensitivity function which is given by
\[
S(s) = \frac{1}{1 + L(s)} \tag{1.28}
\]
where $L(s)$ is the loop transfer function. In the following we will study both $S(s)$ and the tracking ratio $T(s)$ which is given by
\[
T(s) = \frac{L(s)}{1 + L(s)} = \frac{y(s)}{y_{SP}(s)} \tag{1.29}
\]

### 1.4.2 Frequency response analysis of setpoint tracking

From (1.27) we see we that the response in the control error due to the setpoint is
\[
e_{SP}(s) = S(s)y_{SP}(s) \tag{1.30}
\]

By plotting the frequency response $S(j\omega)$ we can easily calculate how large the error is for a given frequency component in the setpoint: Assume that the setpoint is a sinusoid of amplitude $Y_{SP}$ and frequency $\omega$. Then the steady-state response in the error is
\[
e_{SP}(t) = Y_{SP} |S(j\omega)| \sin[\omega t + \text{arg} S(j\omega)] \tag{1.31}
\]

Thus, the error is small and consequently the tracking property is good if $|S(j\omega)| \ll 1$, while the error is large and the tracking property poor if $|S(j\omega)| \approx 1$.

The tracking property can be indicated by the tracking function $T(s)$, too. The response in the process output due to the setpoint is
\[
y(s) = T(s)y_{SP}(s) \tag{1.32}
\]

Assume that the setpoint is a sinusoid of amplitude $Y_{SP}$ and frequency $\omega$. Then the steady-state response in the process output due to the setpoint is
\[
y(t) = Y_{SP} |T(j\omega)| \sin[\omega t + \text{arg} T(j\omega)] \tag{1.33}
\]

Thus, $|T(j\omega)| \approx 1$ indicates that the control system has good tracking property, while $|T(j\omega)| \ll 1$ indicates poor tracking property.

Since both $S(s)$ and $T(s)$ are functions of the loop transfer function $L(s)$, cf. (1.28) and (1.29), there is a relation between $L(s)$ and the tracking property of the control system. Using (1.28) and (1.28) we can conclude as follows:

Good setpoint tracking: $|S(j\omega)| \ll 1$, $|T(j\omega)| \approx 1$, $|L(j\omega)| \gg 1$ \tag{1.34}
Poor setpoint tracking: \[ |S(j\omega)| \approx 1, \quad |T(j\omega)| \ll 1, \quad |L(j\omega)| \ll 1 \quad (1.35) \]

Figure 1.3 shows typical Bode plots of \(|S(j\omega)|, |T(j\omega)|\) and \(|L(j\omega)|\). Usually we are interested in the amplitude gains, not the phase lags. Therefore plots of \(\arg S(j\omega), \arg T(j\omega)\) and \(\arg L(j\omega)\) are not shown nor discussed here. The bandwidths indicated in the figure are defined below.

The bandwidth of a control system is the frequency which divides the frequency range of good tracking and poor tracking. From (1.34) and (1.35) and Figure 1.3 we can list the following three candidates for a definition of the bandwidth:

- \(\omega_t\), which is the frequency where the amplitude gain of the tracking function has value \(1/\sqrt{2} \approx 0.71 = -3\) dB. This definition is in accordance with the usual bandwidth definition of lowpass filters. The \(\omega_t\) bandwidth is also called the \(-3\) dB bandwidth \(\omega_{-3\text{dB}}\).

- \(\omega_c\), which is the frequency where the amplitude gain of the loop transfer function has value \(1 = -0\) dB. \(\omega_c\) is called the \textit{crossover} frequency of \(L\).

- \(\omega_s\), which is the frequency where the amplitude gain of the sensitivity function has value \(1 - 1/\sqrt{2} \approx 1 - 0.71 \approx 0.29 \approx -11\) dB. This
definition is derived from the \(-3\, \text{dB} \) bandwidth of the tracking function: Good tracking corresponds to tracking gain between \(1/\sqrt{2}\) and 1. Now recall that the sensitivity function is the transfer function from setpoint to control error, cf. (1.30). Expressed in terms of the control error, we can say that good tracking corresponds to sensitivity gain \(|S|\) less than \(1 - 1/\sqrt{2} \approx -11\, \text{dB} \approx 0.29\). The frequency where \(|S|\) is \(-11\, \text{dB}\) is denoted the sensitivity bandwidth, \(\omega_s\).

Of the three bandwidth candidates defined above the sensitivity bandwidth \(\omega_s\) is most closely related to the control error. Therefore \(\omega_s\) may be claimed to be the most convenient bandwidth definition as far as the tracking property of a control system concerns. In addition \(\omega_s\) is a convenient bandwidth related to the compensation property of a control system (this will be discussed in more detail soon). However, the crossover frequency \(\omega_c\) and the \(-3\, \text{dB} \) bandwidth are the commonly used bandwidth definitions.

As indicated in Figure 1.3 the numerical values of the various bandwidth definitions are different (this is demonstrated in Example 1.1).

If you need a (possibly rough) estimate of the response time \(T_r\) of a control system, which is time it takes for a step response to reach 63% of its steady-state value, you can use

\[
T_r \approx \frac{k}{\omega_t} \, [\text{s}] \quad (1.36)
\]

where \(\omega_t\) is the \(-3\, \text{dB} \) bandwidth in rad/s.\(^2\) \(k\) can be set to some value between 1.5 and 2.0, say 2.0 if you want to be conservative.

**Example 1.1 Frequency response analysis of setpoint tracking**

See the block diagram in Figure 1.2. Assume the following transfer functions:

PID controller:

\[
H_c(s) = K_p \left( 1 + \frac{1}{T_i s} + \frac{T_d s}{T_f s + 1} \right) \quad (1.37)
\]

Process transfer functions (second order with time delay):

\[
H_u(s) = \frac{K_u}{(T_1 s + 1)(T_2 s + 1)} e^{-\tau s} \quad (1.38)
\]

\(^2\)How can you find the exact value of the response time? Simulate!
Figure 1.4: Example 1.1: Simulated responses of the control system. The setpoint $y_{SP}$ is sinusoid of frequency $\omega_1 = 0.55 \text{ rad/s}$.

$$H_v(s) = \frac{K_v}{(T_1s + 1)(T_2s + 1)}e^{-\tau s}$$  \hspace{1cm} (1.39)$$

Sensor with scaling:

$$H_s(s) = K_s$$  \hspace{1cm} (1.40)$$

The parameter values are $K_p = 4.3$, $T_i = 1.40$, $T_d = 0.35$, $T_f = 0.1T_d = 0.035$, $K_a = 1$, $K_d = 1$, $T_1 = 2$, $T_2 = 0.5$, $\tau = 0.4$, $K_s = 1$. (The PID parameter values are calculated using the Ziegler-Nichols’ closed loop method.) The operation point is at setpoint value 50%, with disturbance $v = 10\%$ (constant), and nominal control signal = 40%.

Figure 1.4 shows simulated responses in the process output $y$ and in the the control error $e = y_{SP} - y$ when the setpoint $y_{SP}$ is a sinusoid of amplitude 10% (about a bias of 50%) and frequency $\omega_1 = 0.55 \text{ rad/s}$. The frequency of the sinusoidal is chosen equal to the sensitivity bandwidth $\omega_s$. The amplitude of the control error should be $0.29 \cdot 10\% = 2.9\%$, and this is actually in accordance with the simulation, see Figure 1.4.

Figure 1.5 shows Bode plots of $|S(j\omega)|$, $|T(j\omega)|$ and $|L(j\omega)|$.

Let us compare the various bandwidth definitions. From Figure 1.5 we find

- 3 dB bandwidth: $\omega_l = 3.8 \text{ rad/s}$
- Crossover frequency: $\omega_c = 1.7 \text{ rad/s}$
Figure 1.5: Example 1.1: Bode plots of $|L(j\omega)|$, $|T(j\omega)|$ and $|S(j\omega)|$

- Sensitivity bandwidth: $\omega_s = 0.55$ rad/s

These values are actually quite different. (As commented in the text above this example, it can be argued that the $\omega_s$ bandwidth gives the most expressive measure of the control system dynamics.)

Finally, let us read off the response time $T_r$. Figure 1.6 shows the response in $y$ due to a step in $y_{SP}$. From the simulation we read off $T_r \approx 1.1$ s. The estimate (1.36) with $k = 2$ gives $T_r \approx 2/\omega_t = 2/3.8 = 0.53$, which is about half the value of the real (simulated) value.

Figure 1.6: Example 1.1: Step response in process output $y$ after a step in setpoint $y_{SP}$
(1.27) gives the response in the control error due to the disturbance. It is repeated here:

\[ e_v(s) = -S(s)H_v(s)v(s) \]  

(1.41)

Thus, the sensitivity function \( S(s) \) is a factor in the transfer function from \( v \) to \( e \) for the control system. However, \( S(s) \) has an additional meaning related to the compensation of a disturbance, namely it expresses the degree of the reduction of the control error due to using closed loop control. With feedback (i.e. closed loop system) the response in the control error due to the disturbance is \( e_v(s) = -S(s)H_v(s)v(s) \). Without feedback (open loop) this response is \( e_v(s) = -H_v(s)v(s) \). The ratio between these responses is

\[ \frac{e_v(s)\text{ with feedback}}{e_v(s)\text{ without feedback}} = \frac{-S(s)H_v(s)v(s)}{-H_v(s)v(s)} = S(s) \]  

(1.42)

Assuming that the disturbance is sinusoidal with frequency \( \omega \) rad/s, (1.42) with \( s = j\omega \), that is \( S(j\omega) \), expresses the ratio between sinusoidal responses.

Again, effective control, which here means effective disturbance compensation, corresponds to a small value of \( |S| \) (value zero or close to zero), while ineffective control corresponds to \(|S| \) close to or greater than 1. We can define the bandwidth of the control system with respect to its compensation property. Here are two alternate bandwidth definitions:

- The bandwidth \( \omega_s \) – the sensitivity bandwidth – is the upper limit of the frequency range of effective compensation. One possible definition is
  \[ |S(j\omega_s)| \approx 0.29 \approx -11 \text{ dB} \]  
  (1.43)
  which means that the amplitude of the error with feedback control is less than 29% of amplitude without feedback control. The number 0.29 is chosen to have the same bandwidth definition regarding disturbance compensation as regarding setpoint tracking, cf. page 8.

- The bandwidth \( \omega_c \) is the crossover frequency of the loop transfer functions \( \omega_c \), that is,
  \[ |L(j\omega_c)| = 0 \text{ dB} \approx 1 \]  
  (1.44)
Note: The feedback does not reduce the control error due to a sinusoidal disturbance if its frequency is above the bandwidth. But still the disturbance may be well attenuated through the (control) system. This attenuation is due to the typical inherent lowpass filtering characteristic of physical systems (processes). Imagine a liquid tank, which attenuates high-frequent temperature variations existing in the inflow fluid temperature or in the environmental temperature. This inherent lowpass filtering is self regulation.

**Example 1.2 Frequency response analysis of disturbance compensation**

This example is based on the control system described in Example 1.1 (page 9).

Figure 1.7 shows simulated responses in the process output $y$ due to a sinusoidal disturbance $v$ of amplitude 10% (with bias 10%) and frequency $\omega_1 = 0.55\text{rad/s}$. This frequency is for illustration purpose chosen equal to the sensitivity bandwidth of the control system, cf. Figure 1.5. The setpoint $y_{SP}$ is 50%. The control error can be read off as the difference between $y_{SP}$ and $y$. In the first 40 seconds of the simulation the PID controller is in manual mode, so the control loop is open. In the following 40 seconds the PID controller is in automatic mode, so the control loop is closed. We clearly see that the feedback control is effective to compensate for the disturbance at this frequency ($0.55\text{rad/s}$). The amplitude of the control error is 6.6 without feedback and 1.9 with feedback. Thus, the ratio between the closed loop error and the open loop error is $1.9/6.6 = 0.29$, which is in accordance with the amplitude of the sensitivity function at this frequency, cf. Figure 1.5.

Figure 1.8 shows the same kind of simulation, but with disturbance frequency $\omega_1 = 1.7\text{rad/s}$, which is higher than the sensitivity bandwidth, which is 0.55 rad/s. From the simulations we see that closed loop control at this relatively high frequency, 1.7 rad/s, does not compensate for the disturbance — actually the open loop works better. This is in accordance with the fact that $|S(j\omega)|$ is greater than 1 at $\omega = 1.7\text{rad/s}$, cf. the Bode plot in Figure 1.5.

Finally, let us compare the simulated responses shown in Figure 1.8 and in Figure 1.4. The amplitude of the control error is less in Figure 1.8, despite the fact that the closed loop or feedback control is not efficient (at frequency 1.7 rad/s). The relatively small amplitude of the control error is due to the self regulation of the process, which means that the disturbance is attenuated through the process, whether the process is controlled or not.
Figure 1.7: Example 1.2: Simulated responses of the control system. The disturbance $v$ is sinusoidal with frequency $\omega_1 = 0.55$ rad/s. The PID-controller is in manual mode (i.e. open loop control) the first 40 seconds, and in automatic mode (closed loop control) thereafter.

In Example 1.2 I did not choose the disturbance frequency, 1.7 rad/s, by random. 1.7 rad/s is actually the loop transfer function crossover frequency of the control system. Thus, the example demonstrates that the crossover frequency may give a poor measure of the performance of the control system. The sensitivity bandwidth is a better measure of the performance.
Figure 1.8: Example 1.2: Simulated responses of the control system. The disturbance $v$ is sinusoidal with frequency $\omega_1 = 1.7$ rad/s. The PID-controller is in manual mode (i.e. open loop control) the first 40 seconds, and in automatic mode (closed loop control) thereafter.